



# OPERATIONS RESEARCH

## Using Duality and Sensitivity Analysis to Interpret Linear Programming Solutions

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A key problem faced by managers is how to allocate scarce resources among activities or projects. Linear programming, or LP, is a method of allocating resources in an optimal way. It is one of the most widely used operations research (OR) tools. It has been used successfully as a decision-making aid in almost all industries and in financial and service organizations.

*Programming* refers to mathematical programming. In this context, it refers to a planning process that allocates resources—labor, materials, machines, and capital—in the best possible (optimal) way so that costs are minimized or profits are maximized.

The *linear* part of the name refers to the following:

- The objective function (i.e., maximization or minimization) can be described by a linear function of the decision variables, that is, a mathematical function involving only the first powers of the variables with no cross products. For example,  $23X_2$  and  $4X_{16}$  are valid decision variable terms, while  $23X_2^2$ ,  $4X_{16}^3$ , and  $(4X_1 \times 2X_1)$  are not. The entire problem can be expressed as straight lines, planes, or similar geometrical figures.
- The constraint set can be expressed as a set of linear equations.

In addition to the linear requirements, nonnegativity conditions state that the variables cannot assume negative values. It is not possible to have negative resources. Without these conditions, it would be mathematically possible to use more resources than are available.

This publication will build on the example used in EM 8719, *Using the Graphical Method to Solve Linear Programs*, and EM 8720, *Using the Simplex Method to Solve Linear Programming Maximization Problems*. EM 8719 introduces the graphical method for solving LP problems. EM 8720 explains how to use the simplex method to solve maximization problems with less-than-or-equal-to ( $\leq$ ) constraints.

The graphical method can be used when only two products are produced (e.g., tables and chairs). It's possible to graph problems with three products,



but three-dimensional graphs quickly become cumbersome, so the simplex method is used to solve these problems. Problems with more than three products cannot be graphed. Most real-world problems contain numerous constraints and potential products. Thus, for all practical purposes, the graphical method is used only to help explain LP solution procedures.

The simplex method is the most common procedure for solving large LP problems. However, large LP problems would take too long to solve by hand, and it's difficult to solve them without making errors. Thus, the simplex method also has limited practicality in the real world.

Fortunately, many commercial LP software products are available to do the work for you. You simply state the objective function and resource constraints, and the software finds the optimal solution.

All of these software products use a revised simplex method. This method uses the same basic calculations as the regular simplex method, but the entire simplex tableau is not calculated at each step. The information needed to move from one feasible solution to another is calculated directly from the original equations, so less computational effort and time is needed. For a detailed description of the revised simplex method, see Ravindran, et al.

To understand the results of computer-generated LP solutions, it helps to understand how the original simplex method works. See Appendices A and B or EM 8720 for a detailed explanation.

In addition to the optimal solution, LP computer programs provide other useful information that can be used for *sensitivity analysis*. This procedure involves looking at how changes in the original LP problem can affect the optimal solution. For example, there may be a change in resource availability, or management may decide to add a new product. We'll discuss sensitivity analysis later in this publication.

First, however, we'll illustrate how to solve an LP problem using a computer program called LINDO.\* In explaining how to interpret the results, we will introduce duality theory and sensitivity analysis.

\**Solver Suite: LINDO, LINGO, WHAT'S BEST. LINDO Systems Inc., Chicago. 382 pp. This product is mentioned as an illustration only. The Oregon State University Extension Service neither endorses this product nor intends to discriminate against products not mentioned.*

**Programming. . .**

"Refers to mathematical programming."

**Sensitivity analysis. . .**

"Involves looking at how changes in the original LP problem can affect the optimal solution."

## An example: Production of wooden tables, chairs, and benches

A manufacturer produces wooden tables, chairs, and benches. The company wishes to allocate resources so as to maximize profits. Because there are three products (tables, chairs, and benches), it's not practical to use the graphical method to find the optimal solution.

Wood and labor are the two resources used to produce tables, chairs, and benches. It takes 30 board feet (bf) of wood and 5 hours of labor to make a table, 20 bf of wood and 10 hours of labor to make a chair, and 25 bf of wood and 7 hours of labor to make a bench. There are 300 bf of wood available and 110 hours of labor available. Unit profit is \$6 for a table, \$8 for a chair, and \$7 for a bench.

Because demand is high for benches, management decides that at least two benches must be made. This introduces a new constraint into the problem.

Let  $X_1$  represent tables,  $X_2$  represent chairs, and  $X_3$  represent benches.  $Z$  represents profit. The LP problem can be formulated as:

|             |                                  |   |
|-------------|----------------------------------|---|
| Maximize:   | $Z = 6X_1 + 8X_2 + 7X_3$         | (objective function: maximize profit)   |
| Subject to: | $30X_1 + 20X_2 + 25X_3 \leq 300$ | (wood constraint: 300 bf available)     |
|             | $5X_1 + 10X_2 + 7X_3 \leq 110$   | (labor constraint: 110 hours available) |
|             | $X_3 \geq 2$                     | (bench demand)                          |
|             | $X_1, X_2, X_3 \geq 0$           | (nonnegativity conditions)              |

The LP problem in EM 8720 included *slack* variables. This new problem also uses *surplus* and *artificial* variables. The computer software will add these variables as needed, so we don't need to worry about them. For a definition of surplus and artificial variables and the hand solution to this problem, see Appendix B.

### Computer solution

In the following example, we will set up the above LP problem for a software solution, using LINDO as an example. Then, we can look at the results and examine the problem in different ways by using sensitivity analysis.



#### Tip . .

In our example,  $X_1$  refers to tables,  $X_2$  refers to chairs,  $X_3$  refers to benches, and  $Z$  refers to profit.

In the EDIT box of LINDO, state the LP as:

```

MAX 6X1 + 8X2 + 7X3
SUBJECT TO
30X1 + 20X2 + 25X3 <= 300
5X1 + 10X2 + 25X3 <= 110
X3 >= 2
    
```

That's it! The software will solve the problem. It will add slack, surplus, and artificial variables when necessary.

After the LP problem is solved, the output appears as:

LP OPTIMUM FOUND AT STEP 4

OBJECTIVE FUNCTION VALUE

1) 98.18182

| VARIABLE | VALUE    | REDUCED COST |
|----------|----------|--------------|
| X1       | 0.000000 | 0.545455     |
| X2       | 5.909091 | 0.000000     |
| X3       | 7.272727 | 0.000000     |

| ROW | SLACK OR SURPLUS | DUAL PRICES |
|-----|------------------|-------------|
| 2)  | 0.000000         | 0.127273    |
| 3)  | 0.000000         | 0.545455    |
| 4)  | 5.272727         | 0.000000    |

NO. ITERATIONS = 4

It took four iterations or pivots to find the optimal solution as \$98.18182. The VALUE column shows the value of each product (tables, chairs, and benches) in the optimal solution. It will be most profitable to produce 0.0  $X_1$  (tables), 5.909091  $X_2$  (chairs), and 7.272727  $X_3$  (benches). Therefore, the optimal solution is:

$$\$6(0) + \$8(5.909091) + \$7(7.272727) = \$98.18$$



The value in the REDUCED COST column can be interpreted in two ways. For  $X_1$  (tables) to be profitable in the optimal solution, its profit would need to increase by \$0.545455. Thus, if the profit for tables increases from \$6 to \$6.545455, tables will appear in the final solution. All products in the final solution automatically have a reduced cost of 0.

The reduced cost also can be interpreted as the penalty you would pay to introduce a product into the solution. In this problem,

if you force one table into the final solution, you will be penalized by \$0.545455. In other words, the optimal solution (profit) will decrease by \$0.545455.

In a model that seeks to minimize cost, the optimal solution would increase by the amount of the penalty. For example, if you force in one unit of a product with a reduced cost of \$10, the optimal solution (total cost) will increase by \$10.

The SLACK OR SURPLUS column tells you how close you are to satisfying a constraint as an equality. For  $\leq$  constraints, this quantity is known as a slack variable. For  $\geq$  constraints, it is known as a surplus variable. If a constraint is satisfied exactly, the slack or surplus value is 0.

The positive value of 5.272727 in row 4 of our example means that the constraint  $X_3 \geq 2$  (bench demand) is 5.272727 units from being satisfied as an equality. Because the optimal solution resulted in production of 7.272727 benches, and the minimum acceptable number of benches is 2, there are 5.272727 more benches than needed to satisfy the constraint  $X_3 \geq 2$  as an equality.

If a slack or surplus value is negative, the solution is infeasible and the LP problem has been formulated incorrectly. By noting which variable has a negative value, you can identify the misformulated constraint.

The DUAL PRICES figure represents the amount the optimal solution would change (increase for a maximization model or decrease for a minimization model) given a unit of increase on the right-hand side of the constraint. For example, the dual price of 0.127273 in row 2 means that by raising the right-hand side of the wood constraint by 1 bf, to 301 bf, we could improve the solution by 0.127273 or about \$0.13.

Dual prices (also known as shadow prices) indicate how much one would be willing to pay for additional units of a resource. In our example, if we could buy a unit of wood for less than \$0.13, it would be worth doing so because it would increase profit by \$0.13. Dual prices are very important in economic sensitivity analysis, as will be discussed in the following section.

Dual prices (also known as shadow prices) indicate how much one would be willing to pay for additional units of a resource.

## Duality and sensitivity analysis

Sensitivity analysis is used to determine how minor variations in an LP problem affect the final solution. Most real-world LP problems are large, and many of the values in the model are estimates

**Opportunity cost. . .**

“Represents the cost of employing a resource for one use instead of another.”

**Duality theory. . .**

“Every primal has an associated LP problem called a *dual*. If the primal LP objective function is maximized, the dual LP objective function is minimized.”

of future conditions. It is important to study what will happen to the optimal solution if other conditions occur. For example, it may be possible to purchase more resources, or management may need to decide whether to add a new product. You can conduct these types of analysis without rerunning the entire LP.

To understand sensitivity analysis, it’s important to understand the concept of *opportunity cost*. Opportunity cost represents the cost of employing a resource for one use instead of another. In other words, if you give up \$0.50 of profit by using a unit of wood to produce a table instead of a chair, the opportunity cost of the wood is \$0.50.

Another part of sensitivity analysis is *duality theory*. An LP problem like the one we solved using LINDO is called a *primal*. Every primal has an associated LP problem called a *dual*. If the primal LP objective function is maximized, the dual LP objective function is minimized. In other words, when the objective of the primal is profit maximization, the objective of the dual is resource minimization at a certain level of profit. Solving the original LP problem also provides a solution to its dual. The dual prices (shadow prices) in our LINDO example come from the optimal solution for the dual problem.

**Constructing a dual LP**

Let’s simplify our primal LP problem to include only tables and chairs. This problem was graphed and solved in EM 8719. It can be formulated as:

|             |                          |                                       |
|-------------|--------------------------|---------------------------------------|
| Maximize:   | $Z = 6X_1 + 8X_2$        | (objective function: maximize profit) |
| Subject to: | $30X_1 + 20X_2 \leq 300$ | (wood constraint: 300 board feet)     |
|             | $5X_1 + 10X_2 \leq 110$  | (labor constraint: 110 hours)         |
|             | $X_1, X_2 \geq 0$        | (nonnegativity conditions)            |

As solved in EM 8719, the maximum value of the objective function for the primal is:

$$Z = \$6(4) + \$8(9) = \$96$$

In other words, the company would produce four tables and nine chairs for a profit of \$96.

The right-hand values of the primal constraints (300 bf and 110 hours) become the coefficients of the dual objective function. For this example, there are 300 board feet of wood available and 110 hours of labor available. Using R to represent cost,  $O_1$  to represent the opportunity cost of a unit of wood, and  $O_2$  to

represent the opportunity cost of a unit of labor, we can rewrite the objective function as:

$$\text{Minimize } R = 300O_1 + 110O_2$$

This equation states that we want to minimize total cost (R), which is equal to 300 units of wood times the opportunity cost of a unit of wood plus 110 units of labor times the opportunity cost of a unit of labor.

To find the dual's constraints, we transpose the coefficients used on the left-hand side of the primal constraints. These coefficients are shown in Table 1.

Table 1.—Primal LP equations and coefficients.

| Equations             | Coefficients |
|-----------------------|--------------|
| $30X_1 + 20X_2 < 300$ | 30 20        |
| $5X_1 + 10X_2 < 110$  | 5 10         |

Place the primal coefficients in a matrix, which we'll call Matrix A:

$$A = \begin{matrix} 30 & 20 \\ 5 & 10 \end{matrix}$$

Now create a new matrix called Matrix  $A^T$  by transposing the values in A so that the first column of A becomes the first row of  $A^T$ . The second column of A becomes the second row of  $A^T$ , and so on, if there are more columns.

$$A^T = \begin{matrix} 30 & 5 \\ 20 & 10 \end{matrix}$$

The values in the rows of  $A^T$  will become the coefficients on the left-hand side of the dual's constraints.

The dual's constants (right-hand side) are obtained from the objective function of the primal ( $Z = 6X_1 + 8X_2$ ). Thus, the constants are 6 and 8.

If the primal is a maximization problem and the constraints are  $\leq$ , the dual is a minimization problem and the constraints become  $\geq$ . Thus, we can write the constraints as:

$$30O_1 + 5O_2 \geq 6$$

$$20O_1 + 10O_2 \geq 8$$

The first constraint states that the amount of wood needed to produce a table (30 bf) multiplied by the opportunity cost of using the wood to make tables ( $O_1$ ) plus the amount of labor needed to



make a table (5 hours) multiplied by the opportunity cost of using the labor to make tables ( $O_2$ ) must be greater than or equal to \$6. The \$6 is the net profit for a table. If the opportunity cost of producing tables is equal to the net profit, tables will be produced. If it is greater than the net profit, no tables will be produced, and the units of wood and labor will be used elsewhere to attain higher profits.

The second constraint can be interpreted in the same manner. The amount of wood needed to produce a chair (20 bf) times the opportunity cost of using the wood to make chairs ( $O_1$ ) plus the amount of labor needed to make a chair (10 hours) times the opportunity cost of using the labor to make chairs ( $O_2$ ) must be greater than or equal to \$8. The \$8 is the net profit for a chair. If the opportunity cost of producing chairs is equal to the net profit, chairs will be produced. If it is greater than the net profit, no chairs will be produced, and the units will be used elsewhere to attain higher profits.

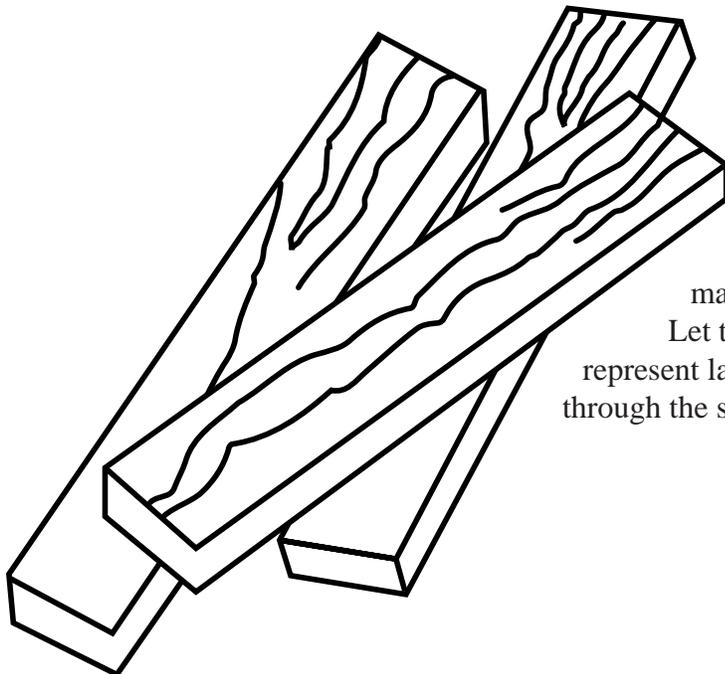
Therefore, we can formulate the dual as:

$$\text{Minimize: } R = 30O_1 + 110O_2$$

$$\text{Subject to: } 30O_1 + 5O_2 \geq 6 \quad (\text{tables})$$

$$20O_1 + 10O_2 \geq 8 \quad (\text{chairs})$$

$$\text{where: } O_1, O_2 \geq 0$$



### Graphing the dual LP

We can graph the dual LP problem. In this example, we won't show every step. For a more detailed explanation of how to solve LP problems graphically, see EM 8719 or one of the sources listed under "For more information."

Let the  $x$  axis represent wood, and the  $y$  axis represent labor. Refer to Figure 1 as we briefly go through the steps of graphically solving the dual.

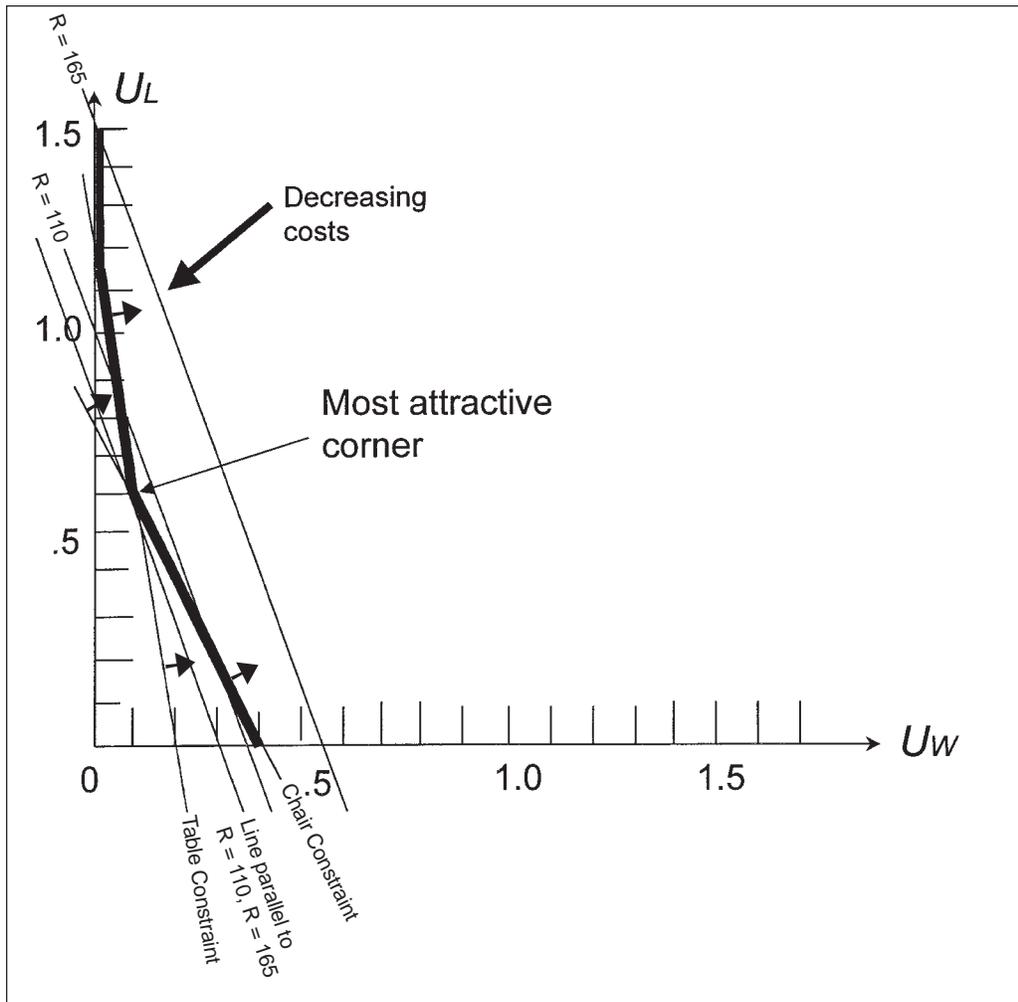


Figure 1.—Table and chair graphical solution.

First, we can determine the two constraint lines:

For the table constraint:      When  $O_1 = 0$ ,  $O_2 = 6/5$  or 1.2

   When  $O_2 = 0$ ,  $O_1 = 6/30$  or 0.2

Plot the table constraint line using the intercepts  $O_1 = 0.2$  and  $O_2 = 1.2$ .

For the chair constraint:      When  $O_1 = 0$ ,  $O_2 = 8/10$  or 0.8

   When  $O_2 = 0$ ,  $O_1 = 8/20$  or 0.4

Plot the chair constraint line using the intercepts  $O_1 = 0.4$  and  $O_2 = 0.8$ .

Next, we can determine the valid side of the constraint lines by substituting the origin (0,0) into the table constraint equation. We get  $30(0) + 5(0) \geq 6$ . This statement is false, so we know the valid side of the table constraint line is away from the origin. In the same

manner,  $20(0) + 10(0) \geq 8$  is false, so the valid side of the chair constraint line also is away from the origin. Place small arrows on both constraint lines pointing away from the origin. Any point on the nonorigin side of the table and chair constraint lines represents a feasible solution. Draw a heavy line to show the boundary of the feasible region.

Plot two objective function lines to determine the direction of improvement.

Let  $R = 165$

$$165 = 300O_1 + 110O_2$$

Set  $O_2 = 0$  and solve for  $O_1$

$$165 = 300O_1 + 110(0)$$

$$165 = 300O_1$$

$$165/300 = O_1$$

$$O_1 = 0.55$$

Let  $R = 110$

$$110 = 300O_1 + 110O_2$$

Set  $O_2 = 0$  and solve for  $O_1$

$$110 = 300O_1 + 110(0)$$

$$110 = 300O_1$$

$$110/300 = O_1$$

$$O_1 = 0.37$$

Next:

Set  $O_1 = 0$  and solve for  $O_2$

$$165 = 300(0) + 110O_2$$

$$165 = 110O_2$$

$$165/110 = O_2$$

$$O_2 = 1.5$$

Set  $O_1 = 0$  and solve for  $O_2$

$$110 = 300(0) + 110O_2$$

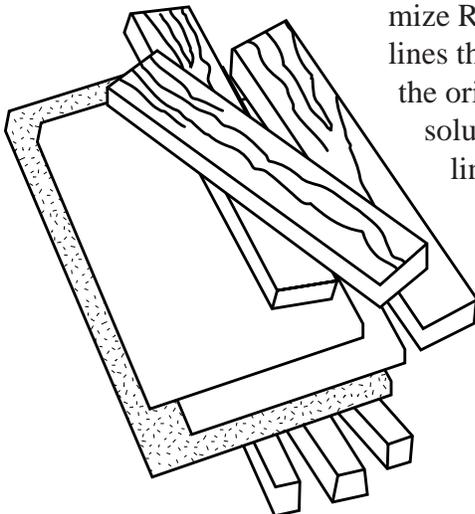
$$110 = 110O_2$$

$$110/110 = O_2$$

$$O_2 = 1$$

We can see from the two objective function lines that  $R$  decreases as we move toward the origin  $(0,0)$ . We want to minimize  $R$ , so we will draw a line parallel to the objective function lines that touches the corner in the feasible region that is closest to the origin. This point is the most attractive corner and the optimal solution to the problem. It is where the table and chair constraint lines intersect.

If our graph were drawn perfectly and we could read it perfectly, we could determine the optimal amounts of tables and chairs from the corresponding coordinates.



Because we cannot read it exactly, we instead will solve the equations simultaneously to determine the exact coordinates:

$$\begin{array}{r} 30O_1 + 5O_2 = 6 \\ -0.5(20O_1 + 10O_2) = 8 \\ \hline 20O_1 + 0O_2 = 2 \end{array}$$

therefore:  $O_1 = 2/20 = (0.10 \text{ or } \$0.10 \text{ per board foot of wood})$

Substitute 0.10 into either of the equations to find the value of  $O_2$ :

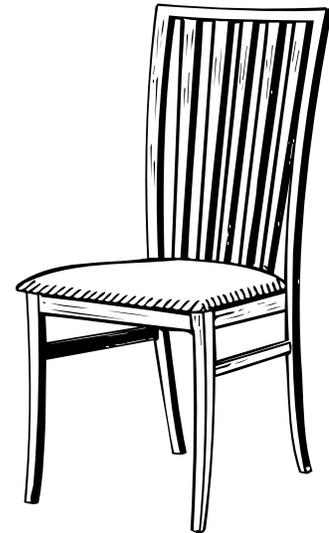
$$30(0.10) + 5O_2 = 6$$

$$3 + 5O_2 = 6$$

$$5O_2 = 6 - 3$$

$$5O_2 = 3$$

$$O_2 = 3/5 = (0.60 \text{ or } \$0.60 \text{ per hour of labor})$$



The final step is to calculate the minimum value of the objective function:

$$R = 300(\$0.10) + 110(\$0.60) = \$96$$

This is the same value that was obtained when maximizing the objective function of the primal.

The \$0.10 represents the actual worth of 1 unit (board foot) of wood to the manufacturing operation. That is, 1 more board foot of wood could be used to produce more furniture sold for a \$0.10 profit. If 1 more board foot of wood were available (301 board feet), total profit would increase by \$0.10. The \$0.10 also can be viewed as the lost potential profit as a result of not having an additional board foot of wood.

Likewise, an additional hour of labor will increase profit by \$0.60. An additional 2 hours of labor will increase profit by 2 x \$0.60 or \$1.20.

These values are the dual prices, or shadow prices, reported by LINDO in our example. They help the company decide whether to buy more wood or labor to increase production. If it can purchase additional wood at \$0.08 per board foot, it will realize a net gain of \$0.02 for each additional board foot of wood acquired. If it can buy an hour of labor for \$0.40, it will realize a net gain of \$0.20 for each additional hour of labor.

**Nonbinding. . .**

“It no longer con-  
strains the optimal  
solution.”

The dual price associated with a particular LP constraint remains constant until the constraint becomes *nonbinding*. In other words, it no longer constrains the optimal solution. For example, adding units of wood at first improves profit. Eventually, however, you will find that adding wood no longer improves profit because you can utilize no more wood without also adding more labor.

When a constraint is nonbinding, the dual price takes on a value of zero. If graphed, the constraint line would lie outside the feasible region. The feasible region is bounded by other constraints, and the optimal solution is found at one or more corners of the feasible region.

Most LP computer programs have a command that lets you view the valid range of dual price values. In LINDO, the command is RANGE.

**Comparing the primal and dual  
simplex tableaus**

Table 2 illustrates the final tableau for solving the primal using the simplex method, and Table 3 shows the final tableau for the dual. For more information on the simplex method, see EM 8720 or Appendices A and B.

Table 2.—Final simplex tableau for the tables and chairs LP primal.

| Unit profit |             | 6     | 8     | 0     | 0     |          |                                 |
|-------------|-------------|-------|-------|-------|-------|----------|---------------------------------|
|             | Basic mix   | $X_1$ | $X_2$ | $S_w$ | $S_L$ | Solution |                                 |
| 6           | $X_1$       | 1     | 0     | 1/20  | -1/10 | 4        | Nonbasic:<br>$S_w = 0, S_L = 0$ |
| 8           | $X_2$       | 0     | 1     | -1/40 | 3/20  | 9        |                                 |
|             | Sacrifice   | 6     | 8     | 1/10  | 6/10  | 96       |                                 |
|             | Improvement | 0     | 0     | -1/10 | -6/10 | —        |                                 |

Table 3.—Final simplex tableau for the tables and chairs LP dual.

| Unit profit |             | 300   | 110   | 0     | 0     |          |                                 |
|-------------|-------------|-------|-------|-------|-------|----------|---------------------------------|
|             | Basic mix   | $O_1$ | $O_2$ | $U_T$ | $U_C$ | Solution |                                 |
| 300         | $O_1$       | 1     | 0     | -1/20 | 1/40  | 0.10     | Nonbasic:<br>$U_T = 0, U_C = 0$ |
| 110         | $O_2$       | 0     | 1     | 1/10  | -3/20 | 0.60     |                                 |
|             | Sacrifice   | 300   | 110   | -4    | -9    | 96       |                                 |
|             | Improvement | 0     | 0     | 4     | 9     | —        |                                 |

To be complete, the dual tableau also would have two artificial variable columns, but they are not shown because they add nothing to the interpretation and it is easier to compare the two tableaus without them.

In the primal tableau (Table 2),  $S_w$  represents unused wood, and  $S_L$  represents unused labor. The  $-1/10$  and  $-6/10$  in the Improvement row show that a unit increase in unused wood decreases profits by \$0.10 per board foot, and a unit increase in unused labor reduces profits by \$0.60 per hour. Increasing the amount of unused resource has the same effect as reducing the available resource. Conversely, if more resource can be made available, profit will increase by \$0.10 for each additional board foot of wood and by \$0.60 for each additional hour of labor.

The 4 under  $U_T$  and the 9 under  $U_C$  in the Improvement row of the dual (Table 3) are the optimal production of tables and chairs found by solving the primal LP. The zero improvements for  $O_1$  and  $O_2$  are the optimal values for the primal slack variables, that is,  $S_w = 0$  and  $S_L = 0$ .

The same exchange coefficients for the nonbasic variable columns are present in each tableau with reversed signs and transposed positions. Thus, the columns of one are the rows of the other.

As you can see from this example, either the primal or the dual can be solved using the simplex method. The other LP problem's final tableau then can be constructed from the solved problem's final tableau.

### Complementary slackness

*Complementary slackness* is the relationship between the slack variables in the primal LP problem and the opportunity costs in the dual. If a primal resource such as wood or labor has positive slack, it is not binding on the optimal solution. Making more of this resource available will not improve profits. At the same time, because all of the resource is not being used, it has an opportunity cost of 0. If we use more of it to make tables or chairs, we do not sacrifice any profit from another use. If, on the other hand, the resource has 0 slack, it is called a *binding constraint*. All of the resource is being used, and making more of it available will improve profit by allowing more production. However, it will have a positive opportunity cost because the additional units will have to be taken from some other use.



#### **Complementary slackness. . .**

"The relationship between the slack variables in the primal LP problem and the opportunity costs in the dual."

#### **Binding constraint. . .**

"A resource with 0 slack. Making more of it available will improve profit by allowing more production."

We can see that multiplying the primal slack times the dual opportunity cost always gives an answer of 0 because one of them always has a value of 0. In the equation below, opportunity costs are called the *dual main*:

$$\text{primal slack} \times \text{dual main} = 0$$

For example, the primal slack variables and their values for the tables and chairs LP are:  $S_w = 0$  (board feet of wood) and  $S_L = 0$  (hours of labor). The dual main costs and their values are  $O_1 = 0.10$  (\$ per board foot of wood) and  $O_2 = 0.60$  (\$ per hour of labor). Because there is no slack, there must be a positive opportunity cost. Therefore:

$$S_w \times O_1 = 0 \times 0.60 = 0 \quad \text{and} \quad S_L \times O_2 = 0 \times 0.60 = 0$$

**Surplus product opportunity cost. . .**

"The opportunity cost of the resources exceeds the unit profit of the product."

**Primal main. . .**

"Production quantities."

**Dual surplus. . .**

"Surplus opportunity costs."

If production is 0 for a given product, it means the resources necessary to make it can be used more profitably elsewhere. In this case, the opportunity cost of the resources exceeds the unit profit of the product so there is *surplus product opportunity cost*. If, on the other hand, the product is produced, then the opportunity costs of the resources must equal its unit profit. The resources are just as valuable when used to make this product as they would be if used elsewhere. Therefore, the surplus product opportunity cost is 0. Again, we can see that multiplying the production quantity of the primal times the surplus opportunity cost of the dual always gives an answer of 0 because one of them always has a value of 0. In the equation below, production quantities are called the *primal main*, and surplus opportunity costs are called the *dual surplus*. Complementary slackness indicates that for every product item:

$$\text{primal main} \times \text{dual surplus} = 0$$

For example, the primal main values for the tables and chairs LP are:  $X_1 = 4$  (tables produced) and  $X_2 = 9$  (chairs produced). The dual surplus values are:  $U_T = 0$  (\$ per table) and  $U_C = 0$  (\$ per chair). Therefore:

$$X_1 \times U_T = 4 \times 0 = 0 \quad \text{and} \quad X_2 \times U_C = 9 \times 0 = 0$$

## Evaluating new products

What if the marketing group suggests adding two more items to the company's product line? Based on a market study, management agrees that enough demand exists to add wooden benches and stools to the product line. The business team calculates that the per-unit profit on benches will be \$7 and the per-unit profit on stools will be \$2. Engineering estimates that a bench will require

25 bf of wood and 7 hours of labor, and a stool will require 10 bf of wood and 2 hours of labor.

Using the dual values for wood and labor, the opportunity cost of a bench and a stool can be calculated as:

$$\text{Opportunity cost of manufacturing a bench} = 25(0.10) + 7(0.60) = \$6.70$$

$$\text{Opportunity cost of manufacturing a stool} = 10(0.10) + 2(0.60) = \$2.20$$

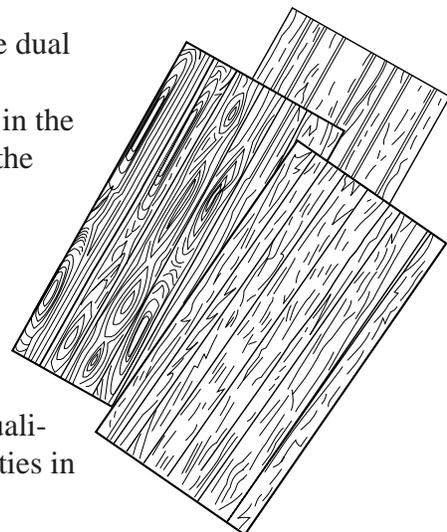
The \$6.70 opportunity cost of a bench is less than the profit of \$7 per bench, so the company should make benches. That is, wood and labor can earn less elsewhere than they can by being used to make benches.

The \$2.20 opportunity cost of a stool is greater than the profit of \$2 per stool, however. Thus, each stool consumes resources in excess of the profit it will earn. Using the resources to make stools is not as profitable as using the resources somewhere else. Thus, the company should not produce stools.

## Summary

It is obvious that solving an LP problem yields much more information than the optimal solution alone. Through sensitivity analysis, we can use the output of the LP solution to help us make production decisions. Much of this information is based on the relationship between the primal LP problem and its dual. For example:

1. The objective function coefficients of the primal LP become the right-hand side constants of the dual. Likewise, the right-hand side constants of the primal are the cost coefficients of the dual's objective function.
2. The constraint inequalities are reversed.
3. The primal's objective function is maximization; that of the dual is minimization.
4. Each column in the primal corresponds to a constraint row in the dual. Therefore, the number of dual constraints is equal to the number of primal variables.
5. Each constraint row in the primal corresponds to a column in the dual. Therefore, there is one dual variable for every primal constraint.
6. The dual of the dual is the primal.
7. For both the primal LP and its dual, the variables are nonnegative, and the constraints are inequalities. The inequalities in a maximization problem must be  $\leq$ , and the inequalities in



a minimization problem must be  $\geq$ . Such problems are called *symmetric dual linear problems*.

As with most modeling exercises, the most important (and usually most difficult) part of using LP for decision making is problem formulation. Expect to spend most of your effort on this step. If data are incorrect and/or the formulation of the LP problem is wrong, LP software will generate a solution that can lead to unprofitable decisions.

## APPENDIX A

### Definition of the simplex method

Simplex is a mathematical term. In one dimension, a simplex is a line segment connecting two points. In two dimensions, a simplex is a triangle formed by joining three points. A three-dimensional simplex is a four-sided pyramid having four corners. The underlying concepts are geometrical, but the solution algorithm, developed by George Dantzig in 1947, is basically an algebraic procedure. As with the graphical method, the simplex method finds the most attractive corner of the feasible region to solve the LP problem. Remember, any LP problem having a solution must have an optimal solution that corresponds to a corner, although there may be multiple, also known as alternative, optimal solutions.

The simplex method examines corners in a systematic manner to find the optimum. For example, a moderately sized LP with 10 products and 10 resource constraints would involve nearly 200,000 corners. An LP problem 10 times this size would have over a trillion corners. The search procedure for the simplex method is efficient enough that only about 20 of the 200,000 corners must be searched to find the optimal solution.

## APPENDIX B

### Hand solution for the tables, chairs, and benches problem

#### *Definition of a standard form problem*

In a standard form maximization LP problem, all constraints are less-than-or-equal-to ( $\leq$ ) constraints, and all variables are nonnegative. In a standard form minimization LP problem, all constraints are greater-than-or-equal-to ( $\geq$ ) constraints, and again all variables are nonnegative.

Because bench demand is represented as a  $\geq$  constraint in our problem, this LP problem does not meet the requirements for a

maximization problem. To convert it to standard form, multiply the constraint by -1. This transforms the bench demand to:

$$-X_3 \leq -2 \quad (\text{bench demand})$$

If an LP problem has an equality constraint such as  $X_4 = 10$ , replace the constraint with two equivalent inequality constraints:

$$X_4 \geq 10$$

$$X_4 \leq 10$$

Next, convert the  $\geq$  constraint into a  $\leq$  constraint by multiplying by -1.

$$-X_4 \leq -10$$

$$X_4 \leq 10$$

The two constraints are equal to, and take the place of, the equality ( $=$ ) constraint. These constraints now meet the requirements for a standard form maximization LP problem.

These rules are important for establishing duality relations. However, when actually solving an LP problem that includes nonstandard constraints, we keep the constraints in their original form and add *surplus* and *artificial* variables.

### **Surplus and artificial variables**

To use the simplex method to solve an LP problem, we must state the inequalities as equalities. In EM 8720, we used slack variables (designated by  $S$ ) to represent unused resources so that we could change the  $\leq$  inequalities to equalities. In our current problem, bench demand is a  $\geq$  inequality. We will use a surplus variable (designated by  $U$ ) to represent the number of benches produced over the two needed to meet demand.

We can represent bench demand algebraically as:

$$X_3 - U_B = 2$$

This equation states that the number of benches produced minus any surplus benches produced must equal 2 benches. Surplus variables, like all simplex variables, must be nonnegative.

Surplus variables cannot be in the initial basic mix. For example, if we zeroed out the  $X_3$  term in the above equation, we would have:

$$0 - U_B = 2 \text{ or } U_B = -2$$

Clearly, we cannot produce -2 benches.

To use the simplex method with  $\geq$  constraints, one more type of variable must be introduced—an artificial variable. The artificial variable has no economic meaning but is used only to satisfy the

#### **Surplus variable. . .**

“Represents the number of pieces produced over those needed to meet demand.”

#### **Artificial variable. . .**

“Has no economic meaning but is used only to satisfy the requirements of the simplex method.”

requirements of the simplex method. We will use a lower-case  $a$  to represent artificial variables.

Using surplus and artificial variables, we now can state bench demand as:

$$X_3 - U_B + a_B = 2$$

Surplus and slack variables contribute nothing to profit.

Surplus and slack variables contribute nothing to profit. Therefore, their coefficients equal 0 in the objective function.

In order not to violate the underlying production requirement, the artificial variable must have a value of 0. This ensures that it does not appear in the final basic mix. To accomplish this, we assign it a very large negative value per unit profit. (For a minimization problem, we would assign it a very large positive value per unit cost.) The exact value is unimportant as long as it is sufficiently large to make the artificial variable unprofitable during problem solving. We'll represent this number by  $-M$  ( $M$  for a minimization problem). This procedure is known as the *big-M method*.

We now are ready to formulate the LP problem using slack, surplus, and artificial variables. As before, we list all variables in all equations to arrive at:

|             |   |                            |
|-------------|---|----------------------------|
| Maximize:   | $Z = 6X_1 + 8X_2 + 7X_3 + 0S_w + 0S_L + 0U_B - Ma_B$      | (objective function)       |
| Subject to: | $30X_1 + 20X_2 + 25X_3 + 1S_w + 0S_L + 0U_B + 0a_B = 300$ | (wood constraint)          |
|             | $5X_1 + 10X_2 + 7X_3 + 0S_w + 1S_L + 0U_B + 0a_B = 110$   | (labor constraint)         |
|             | $0X_1 + 0X_2 + 1X_3 + 0S_w + 0S_L - 1U_B + 1a_B = 2$      | (bench demand)             |
|             | $X_1, X_2, X_3, S_w, S_L, U_B, a_B \geq 0$                | (nonnegativity conditions) |

The initial simplex tableau is shown in Table A-1. A step-by-step procedure for generating the second simplex tableau will be described. The remaining simplex tableaus will be shown, and an explanation of the final tableau will follow. For more detail on the simplex method, please see EM 8720 or one of the sources listed under "For more information."

The basic mix consists of the slack variables and the artificial variable. There is a 1 in their row and 0s elsewhere. The unit profit for the basic mix variables is brought down into the unit profit column to the left of the tableau (Table A-1).

Table A-1.—Initial simplex tableau.

| Unit profit |             | 6     | 8     | 7     | 0     | 0     | 0     | -M    |          | Exchange ratios |
|-------------|-------------|-------|-------|-------|-------|-------|-------|-------|----------|-----------------|
|             | Basic mix   | $X_1$ | $X_2$ | $X_3$ | $S_w$ | $S_L$ | $U_D$ | $a_B$ | Solution |                 |
| 0           | $S_w$       | 30    | 20    | 25    | 1     | 0     | 0     | 0     | 300      | $300/25 = 12$   |
| 0           | $S_L$       | 5     | 10    | 7     | 0     | 1     | 0     | 0     | 110      | $110/7 = 15.71$ |
| -M          | $a_B$       | 0     | 0     | ①     | 0     | 0     | -1    | 1     | 2        | $2/1 = 2$       |
|             | Sacrifice   | 0     | 0     | -M    | 0     | 0     | M     | -M    | -2M      |                 |
|             | Improvement | 6     | 8     | $7+M$ | 0     | 0     | -M    | 0     | —        |                 |

To calculate the sacrifice row, multiply each basic mix unit profit by each number in its tableau row. Then add the answers for each column. For example, the sacrifice value for the  $X_1$  column is calculated as:

$$\begin{array}{r}
 0 \times 30 = 0 \quad \text{and for the } X_3 \text{ column:} \quad 0 \times 25 = 0 \\
 0 \times 5 = 0 \quad 0 \times 7 = 0 \\
 + (-M \times 0) = 0 \quad \quad \quad \quad \quad \quad \quad \quad \quad + (-M \times 1) = -M \\
 \hline
 0 \quad -M
 \end{array}$$

Do this for each column in the tableau and also for the solution column.

To get the improvement values, subtract the sacrifice value from the unit profit listed at the top of each column. For example, the improvement value for the  $X_1$  column is calculated as:

$$6 - 0 = 6$$

For the  $X_3$  column, it is:

$$7 - (-M) = 7 + M$$

Do this for each column in the tableau except the solution column.

Determine the entering variable by finding the greatest improvement that can be made. Because  $7 + M$  is the largest positive number,  $X_3$  is the entering variable. Mark this column with an arrow pointing down. If no improvement can be found (all of the values in the Improvement row are 0 or a negative number), the optimal solution is represented by the current tableau.

Apply the exit criterion. First, divide each solution value by the value in the same row of the entering variable's column (the  $X_3$  column). If the divisor is 0, the product is infinity, which you can

**Pivot element . . .**

“The value found at the intersection of the entering variable column and the exiting variable row.”

treat as a very large positive number. The resulting values are called *exchange ratios*. They are calculated as:

$$300/25 = 12$$

$$110/7 = 15.71$$

$$2/1 = 2$$

Choose the smallest nonnegative exchange ratio. This represents the exiting variable, which in this case is  $a_B$ . Mark the exiting variable row with an arrow pointing left.

Construct a new simplex tableau. The entering variable,  $X_3$ , replaces the exiting variable,  $a_B$ . Circle the *pivot element*—the value found at the intersection of the entering variable column and the exiting variable row. For the first tableau it is 1, located where the  $X_3$  entering variable column meets the  $a_B$  exiting variable row. Calculate the values for the new entering variable row ( $X_3$ ) in the second tableau by dividing the entries in the old exiting variable row ( $a_B$ ) by the pivot element. Because the pivot element is 1, there is no change, and the values can be brought down into the new row.

The other new row values can be calculated by:

(old value) - (value in that row and in the pivot column X new entering variable value found in the old value column)

For example, in Table A-2, the new  $S_w$  row values are calculated as:

$$30 - 25(0) = 30$$

$$20 - 25(0) = 20$$

$$25 - 25(1) = 0$$

$$1 - 25(0) = 1$$

$$0 - 25(0) = 0$$

$$0 - 25(-1) = 25$$

$$0 - 25(1) = -25$$

Use the same method for calculating the new  $S_L$  row values.

The new tableau is shown in Table A-2. It will take three more simplex tableaus to find the optimal solution. These tableaus are shown in Tables A-3, A-4, and A-5.

Practice using the shortcuts from EM 8720 when constructing the remaining simplex tableaus. As a reminder, they are:

- In the new tableau, only the columns for the nonbasic and exiting variables change. Move the values for all other basic variables directly into the new tableau.
- When 0 is found in the pivot column, that row always is the same in the new tableau. When 0 is found in the pivot row, that column always is the same in the new tableau.
- With two exceptions, the newly entered basic variable's column will contain a 0 in all locations. A 1 will go in the same location as the pivot element in the preceding tableau, and the per-unit profit or loss for this variable will appear in the sacrifice row.

Table A-2.—Second simplex tableau.

| Unit profit |             | 6     | 8       | 7     | 0     | 0     | 0     | -M    |          | Exchange ratios |
|-------------|-------------|-------|---------|-------|-------|-------|-------|-------|----------|-----------------|
|             | Basic mix   | $X_1$ | $X_2$ ↓ | $X_3$ | $S_w$ | $S_L$ | $U_D$ | $a_B$ | Solution |                 |
| 0           | $S_w$       | 30    | 20      | 0     | 1     | 0     | 25    | -25   | 250      | 50/20 = 12.25   |
| 0           | $S_L$ ←     | 5     | 10      | 0     | 0     | 1     | 7     | -7    | 96       | 96/10 = 9.6     |
| 7           | $X_3$       | 0     | 0       | 1     | 0     | 0     | -1    | 1     | 2        | 2/0 = ∞         |
|             | Sacrifice   | 0     | 0       | 7     | 0     | 0     | -7    | 7     | 14       |                 |
|             | Improvement | 6     | 8       | 0     | 0     | 0     | 7     | -M-7  | —        |                 |

Table A-3.—Third simplex tableau.

| Unit profit |             | 6       | 8     | 7     | 0     | 0     | 0     | -M     |          | Exchange ratios |
|-------------|-------------|---------|-------|-------|-------|-------|-------|--------|----------|-----------------|
|             | Basic mix   | $X_1$ ↓ | $X_2$ | $X_3$ | $S_w$ | $S_L$ | $U_D$ | $a_B$  | Solution |                 |
| 0           | $S_w$ ←     | 20      | 0     | 0     | 1     | -2    | 11    | -11    | 58.0     | 58/20 = 2.9     |
| 8           | $X_2$       | 1/2     | 1     | 0     | 0     | 1/10  | 7/10  | -7/10  | 9.6      | 9.6/.5 = 19.2   |
| 7           | $X_3$       | 0       | 0     | 1     | 0     | 0     | -1    | 1      | 2.0      | 2/0 = ∞         |
|             | Sacrifice   | 4       | 8     | 7     | 0     | 0.8   | -1.4  | 1.4    | 90.8     |                 |
|             | Improvement | 2       | 0     | 0     | 0     | -.8   | 1.4   | -M-1.4 | —        |                 |

Table A-4.—Fourth simplex tableau.

| Unit profit |             | 6     | 8     | 7     | 0     | 0     | 0     | -M     |          | Exchange ratios      |
|-------------|-------------|-------|-------|-------|-------|-------|-------|--------|----------|----------------------|
|             | Basic mix   | $X_1$ | $X_2$ | $X_3$ | $S_w$ | $S_L$ | $U_D$ | $a_B$  | Solution |                      |
| 6           | $X_1$ ←     | 1     | 0     | 0     | 1/20  | -1/10 | 11/20 | -11/20 | 2.9      | 2.9/(11/20) = 5.27   |
| 8           | $X_2$       | 0     | 1     | 0     | -1/40 | 3/20  | 17/40 | -17/40 | 8.15     | 8.15/(17/40) = 19.18 |
| 7           | $X_3$       | 0     | 0     | 1     | 0     | 0     | -1    | 1      | 2.0      | 2/-1 = -2            |
|             | Sacrifice   | 6     | 8     | 7     | 0.1   | 0.6   | -0.3  | 0.3    | 96.6     |                      |
|             | Improvement | 0     | 0     | 0     | -0.1  | -0.6  | 0.3   | -M-0.3 | —        |                      |

Table A-5.—Final simplex tableau.

| Unit profit |             | 6      | 8     | 7     | 0      | 0     | 0     | -M    |          | Exchange ratios |
|-------------|-------------|--------|-------|-------|--------|-------|-------|-------|----------|-----------------|
|             | Basic mix   | $X_1$  | $X_2$ | $X_3$ | $S_w$  | $S_L$ | $U_D$ | $a_B$ | Solution |                 |
| 0           | $U_D$       | 20/11  | 0     | 0     | 1/11   | -2/11 | 1     | -1    | 58/11    | 58/11           |
| 8           | $X_2$       | -17/22 | 1     | 0     | -7/110 | 5/22  | 0     | 0     | 65/11    | 65/11           |
| 7           | $X_3$       | 20/11  | 0     | 1     | 1/11   | -2/11 | 0     | 0     | 80/11    | 80/11           |
|             | Sacrifice   | 72/11  | 8     | 7     | 7/55   | 6/11  | 0     | 0     | 1080/11  |                 |
|             | Improvement | -6/11  | 0     | 0     | -7/55  | -6/11 | 0     | -M    | —        |                 |

In Table A-5, there is no positive number in the Improvement row. Thus, no improvement can be made, and this tableau represents the optimal solution to the LP problem. The optimal solution is:

$$\begin{aligned}
 X_1 &= 0 & S_w &= 0 \\
 X_2 &= 65/11 = 5.9 & S_L &= 0 \\
 X_3 &= 80/11 = 7.3 & U_D &= 58/11 = 5.3
 \end{aligned}$$

The company should make 0 tables, 5.9 chairs, and 7.3 benches. Total profit will be \$98.30:

$$6(0) + 8(5.9) + 7(7.3) = \$98.30$$

The total profit we found by hand is slightly different than the profit of \$98.18 found by LINDO because of rounding in the hand solution.

The optimal solution shows that it no longer is profitable to make tables while making benches. The surplus variable is greater than 0;  $U_D = X_3 - 2 = 5.3$ . Thus, 5.3 more benches should be produced than the minimum of 2.

There is no guarantee of an integer (whole number) solution. Obviously, the company cannot sell 0.9 of a chair or 0.3 of a bench. We can think of this fractional part as work in progress or we can round up or down. An optimal solution for another problem might be to buy 6.75 trucks. Obviously, you can buy either 6 or 7 trucks, but not 0.75 of a truck. In this case, rounding up (and for many noninteger answers, rounding either up or down) offers a practical solution. However, if you're studying the possibility of building a new plant addition, a variable can have only a value of 0 or 1, so a fractional answer is useless. Either you build the addition or you don't. Fortunately, methods called *integer programming* can handle these types of problems.

There is no guarantee of an integer (whole number) solution.

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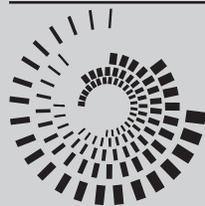
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